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ABSTRACT

The amalgamation class Amal($\mathcal{N}$) of a lattice variety generated by a pentagon is considered. It is shown that Amal($\mathcal{N}$) is closed under reduced products and therefore is an elementary class determined by Horn sentences. The above result is based on a new characterization of Amal($\mathcal{N}$). The lattice varieties whose amalgamation classes contain Amal($\mathcal{N}$) as a subclass are considered.

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INTRODUCTION

We consider the amalgamation class Amal($\mathscr{N}$) of a lattice variety generated by the pentagon. The central result of the paper is Theorem 3.1, that Amal($\mathscr{N}$) is an elementary class. It is closed under reduced products and therefore is determined by Horn sentences Furthermore, if B is a subdirect power of the pentagon and also an image of $A \in$ Amal($\mathscr{N}$), then $B \in$ Amal($\mathscr{N}$).

This result contrasts with the following of Bergman (see [2] Theorem 2): If $\mathscr{V}$ is a variety generated by a finite modular lattice then Amal($\mathscr{V}$) is not an elementary class.

The proof of Theorem 3.1 is based on Theorem 2.2 which gives a new characterization of membership of Amal($\mathscr{N}$). The first characterization was given by Bergman (see [9] Theorem 6.2).

The examples of lattice varieties whose amalgamation class contains Amal($\mathscr{N}$) as a subclass are considered in Proposition 4.1.

0. PRELIMINARIES

0.1 Notation, Terminology and some required results.

0.1.1 Let u/v be a quotient of a lattice L. We shall say that u/v is an N–quotient of L if, for some $z \in L$ the set {u,v,z} generates a sublattice of L isomorphic to a pentagon N (see Figure 1) with u/v as a critical quotient of this pentagon. In this case we write N(u/v, z). (Note that c/a is a critical quotient of N in Figure 1).

0.1.2 We shall use $\underline{1}$, $\underline{2}$, and $\underline{3}$ to denote one–, two– and three–element chains respectively. If $X \cong \underline{2}$ ($X \cong \underline{3}$) and X is a sublattice of a lattice L we say that X is a $\underline{2}$–($\underline{3}$–) sublattice of L if there is a retraction of L onto X.

Corollary Let $\{p>q\} \subseteq N$ and assume that $\{p,q\} \neq \{c,a\}$. If K is any sublattice of N which contains {p,q}, then {p,q} is a $\underline{2}$–subset of K.

Figure 1

0.1.3 Con(A) denotes the congruence lattice of an algebra A. The smallest congruence on A which identifies $a, b \in A$ is denoted Con(a,b).

0.1.4 Let $f\colon A \to B$ be an epimorphism. The kernel of f, kerf, is called a B–congruence on A.

0.1.5 Let $A = \prod_{i \in I} A_i$ be a product of algebras. The congruence φ on A is said to be filtral if it is induced by an ultrafilter on I. It is well–known that if $\{A_i\colon i \in I\}$ is a finite set of finite algebras then every filtral congruence is an A_i –congruence for some $i \in I$.

Let A be congruence distributive. We shall frequently use the following fundamental result:

Jónsson's Lemma: If B is a subalgebra of A and $\theta \in$ Con(B) is completely meet irreducible then there is a filtral congruence φ on A such that $\varphi|_B \subset \theta$.

0.1.6 Let A be as in 0.1.5. A subalgebra B of A is regular if for any two kernels θ_i, θ_j of distinct projections on A we have $\theta_i|_B \neq \theta_j|_B$.

0.1.7 The variety generated by N is denoted $\mathcal{N}$. The last two statements of this subsection are concerned with members of $\mathcal{N}$.

0.1.8 Let ψ be a congruence on a lattice L. Call ψ distributive if L/ψ is a distributive lattice.

Corollary Let $L \in \mathcal{N}$. Then $\psi \in \mathrm{Con}(L)$ is the smallest distributive congruence on L if and only if

$$\psi = \cap\, \theta_i = \Sigma\, \mathrm{Con}(u_s/v_s)$$

where $\{\theta_i\colon\ i \in I\}$ is the set of all $\underline{2}$–congruences on L and $\{u_s/v_s :\ s \in S\}$ is the set of all N–quotients of L (certainly if L is distributive $S = \varphi$).

0.1.9 Let A be a subdirect power of N and let p/q be a non–trivial quotient of A. Then for some N–congruence φ on A we have $(p,q) \notin \varphi$ so that $(p/\varphi)/(q/\varphi)$ is a non–trivial quotient of a pentagon A/φ. Since the pentagon is a projective lattice, there is an N–quotient u/v of A such that $(u/\varphi)/(v/\varphi)$ is a critical quotient of the pentagon A/φ. Since $(u/\varphi)/(v/\varphi)$ projects weakly onto $(p/\varphi)/(v/\varphi)$ in A/φ it follows that there is a quotient u'/v' of A which projects weakly onto both u/v and p/q in at most five steps (see [11]). Since $A \in \mathcal{N}$ we have that u'/v' is an N–quotient (see [10]). Thus A is a subdirect power of N if and only if for any non–trivial quotient p/q of A there is an N–quotient u/v of A which projects weakly onto p/q in at most five steps.

Although the following result is known we shall prove it to make the paper self–contained.

Lemma Let B be a reduced product of the family $\{A_i\colon\ i \in I\}$ and assume that each A_i is a subdirect power of N. Then B is a subdirect power of N.

Proof: Let $A = \Pi\, A_i$ and $\psi \in \mathrm{Con}(A)$ such that $A/\psi = B$. Let $\mathcal{D}$ be a filter on I which induces ψ. Let $(p/\psi)/(q/\psi)$ be a non–trivial quotient of B $(p,q \in A)$. Then

$$X = \{i \in I:\ p_i > q_i\} \in \mathscr{D}$$

Since each A_i is a subdirect power of N for any $i \in X$ there is an N–quotient u_i/v_i of A_i such that u_i/v_i projects weakly onto p_i/q_i in at most five steps. Consider a quotient u/v of A which is described as follows:

If $i \in X$ then u_i/v_i is an N–quotient which projects onto p_i/q_i.

If $i \notin X$ then $u_i = v_i$.

Then u/v is an N–quotient which projects weakly on p/q in A so that $(u/\psi)/(v/\psi)$ is an N–quotient in B which projects weakly onto $(p/\psi)/(q/\psi)$.

0.2 Absolute Retracts and Amalgamation in some congruence distributive varieties.

By a diagram in a variety $\mathscr{V}$ we mean a quintuple (A, f, B, g, C) with A, B, C $\in \mathscr{V}$ and embeddings f: A → B, g: A → C. By an amalgam in $\mathscr{V}$ of this diagram we mean a triple (D, f_1, g_1) with D $\in \mathscr{V}$ and f_1: B → D, g_1: C → D embeddings such that $f f_1 = g g_1$. If such an amalgam exists we say that the diagram can be amalgamated in $\mathscr{V}$. An algebra A $\in \mathscr{V}$ is called an amalgamation base for $\mathscr{V}$ if every diagram (A, f, B, g, C) can be amalgamated in $\mathscr{V}$. The class of all amalgamation bases for $\mathscr{V}$ is called the amalgamation class and is denoted Amal($\mathscr{V}$).

It is known that Amal($\mathscr{V}$) is a proper class ([12]).

An algebra A $\in \mathscr{V}$ is said to be an absolute retract in $\mathscr{V}$ if for any embedding f: A → B $\in \mathscr{V}$ there is an epimorphism g: B → A such that gf is an identity map on A. Absolute retracts of a variety $\mathscr{V}$ belong to Amal($\mathscr{V}$) (see[1] Proposition 3.2).

Example 0.2.1. Let $\mathscr{V}$ be a residually small, congruence distributive variety. Then the collection of subdirectly irreducible members of $\mathscr{V}$ forms a set (up to equivalence).

The subdirectly irreducible members which have no essential extensions are absolute retracts in $\mathscr{V}$ (see [1]). We call them maximal subdirectly irreducible members of $\mathscr{V}$. The set of maximal subdirectly irreducible members of $\mathscr{V}$ is denoted by $\mathscr{V}_{MI}$.

For instance if $\mathscr{N}$ is the variety generated by a pentagon then $\mathscr{N}_{MI} = \{N\}$. Similarly if $\mathscr{L}$ is the variety generated by any lattice L then L is the only absolute retract of $\mathscr{L}$ (see Figures 2 & 3).

The proof of the following result may be found in [3] and [7].

Theorem 0.2.2. Let $\mathscr{V}$ be a residually small, congruence distributive variety and assume that every member of $\mathscr{V}$ has a one–element subalgebra. Then

(1) Amal($\mathscr{V}$) is closed under finite products.

(2) A $\in$ Amal($\mathscr{V}$) iff for any embedding f: A → B $\in \mathscr{V}$ and any homomorphism g: A → M $\in \mathscr{V}_{MI}$ there is a homomorphism h: B → M such that g = hf.

(3) A product of absolute retracts of $\mathscr{V}$ is an absolute retract and therefore belongs to Amal($\mathscr{V}$).

Example 0.2.3. (1) Let L be one of the lattices of Figures 1, 2 or 3 and let $\mathscr{V}$ be a variety generated by L. Then for an arbitrary index set I the direct power L^I is an absolute retract in $\mathscr{V}$ and therefore belongs to Amal($\mathscr{V}$).

(2) We show later (Proposition 4.1) that if $L = L_1, L_2$ or $L = L_{15}^{n}$ for some $n \in \omega$ then Amal($\mathscr{N}$) $\subset$ Amal($\mathscr{V}$). On the other hand, if $L \in \{L_3, \ldots, L_{14}\}$, then Amal($\mathscr{N}$) is not contained in Amal($\mathscr{V}$). This can easily be seen from Theorem 0.2.2 (2): consider embeddings f and g of N into L such that $f(b) = b_1$ and $g(b) = b_2$. Then there is no h: L → L such that g = hf.

The proof of the following lemma can be found in [6].

Lemma 0.2.4. Let $\mathcal{V}$ be an arbitrary variety. A diagram (A, f_0, B_0, f_1, B_1) can be amalgamated in $\mathcal{V}$ iff for $\{i, j\} = \{0, 1\}$ and $a \neq b \in B_i$ there is an algebra $D \in \mathcal{V}$ and homomorphisms $g : B_i \to D$ and $h : B_j \to D$ such that $gf_i = hf_j$ and $g(a) \neq g(b)$.

1. CONGRUENCES & HOMOMORPHISMS

The results of this section are mainly technical. They are needed to prove Theorems 2.2 and 3.1.

1.1 Finitely generated congruence distributive varieties.

Lemma 1.1.1. Let $\mathcal{V}$ be a congruence distributive variety generated by a finite set $\{M_1, \ldots, M_n\}$ of its finite maximal subdirectly irreducible members. Suppose further that, for distinct $i, j \in \{1, \ldots, n\}$, M_i is not an image of a subalgebra of M_j. Then if $A \leq B \in \mathcal{V}$, every M_i–congruence on A can be extended to an M_i–congruence on B.

Proof: Since we have $B \leq M_{s_1}^{I_1} \times \ldots \times M_{s_k}^{I_k}$ where $s_1, \ldots, s_k \in \{1, \ldots, n\}$, it suffices to prove the lemma under the assumption that $B = M_{s_1}^{I_1} \times \ldots \times M_{s_k}^{I_k}$. Let θ be an M_i–congruence on A. By Jónsson's Lemma there is a filtral congruence $\psi \in \mathrm{Con}(B)$ such that $\psi|_A \subset \theta$. Since the set $\{s_1, \ldots, s_k\}$ is finite and each $M_{s_1}, \ldots, M_{s_k}$ is also finite, we have $B/\psi \cong M_{s_t}$ for some $t \in \{1, \ldots, k\}$. It follows from the second assumption of the lemma that $\psi|_A = \theta$ so that $M_{s_t} = M_i$.

Corollary 1.1.2. Let $\mathcal{V}$ be as in Lemma 1.1.1 and let B and C be products of members of $\{M_1, \ldots, M_n\}$. If $f\colon A \to B$ and $g\colon A \to C$ are subdirect representations then the diagram (A, f, B, g, C) can be amalgamated in $\mathcal{V}$.

Proof: We use Lemma 0.2.4. Let $u \neq v \in B$ so that $\pi(u) \neq \pi(v)$ for some projection π on B. Since f is a subdirect representation the map πf is an epimorphism of A onto some member of $\{M_1, \ldots, M_n\}$. It follows from Lemma 1.1.1 that there is a map h on C such that $hg = \pi f$.

Corollary 1.1.3. Let $\mathcal{V}$ be as in Lemma 1.1.1 and assume that A is a subdirect product of members of $\mathcal{V}_{MI}$. Then $A \in \mathrm{Amal}(\mathcal{V})$ iff for any regular sudirect representation $f\colon A \to B$ and any homomorphism $k\colon A \to M \in \mathcal{V}_{MI}$ there is a homomorphism $h : B \to M$ such that $k = hf$.

Proof: The necessity follows from Theorem 0.2.2 (2). To prove that the conditions are sufficient we have to show that if B, C are products of members of $\mathcal{V}_{MI}$ and $f\colon A \to B$, $g\colon A \to C$ are embeddings, then the diagram (A, f, B, g, C) can be amalgamated in $\mathcal{V}$. (This is because every member of $\mathcal{V}$ is embeddable into a product of members of $\mathcal{V}_{MI}$.)

Now since A is a subdirect product of members of $\mathcal{V}_{MI}$, the embeddings f and g induce regular subdirect representations $f'\colon A \to B' \leq B$ and $g'\colon A \to C' \leq B$ (see Figure 4). By Corollary 1.1.2 the diagram (A,f',B',g',C') has an amalgam H. We may assume $H \in \mathrm{Amal}(\mathcal{V})$. To complete the proof it is sufficient to show that the diagrams (A, f, B, f', B') and (A, g, C, g', C') have amalgams (D, s_1, t) and (E, l_1, m). Indeed, if such amalgams exist then, since B', $C' \in \mathrm{Amal}(\mathcal{V})$ by Theorem 0.2.2 (3)), the diagrams (B', q, H, t, D) and (C, m, E, p, H) will have amalgams (F, v, s_2) and (G, u, l_2).

Figure 4

Further, since $H \in \mathrm{Amal}(\mathscr{V})$, the diagram (H, v, F, u, G) will have an amalgam (X, s_3, t_3) so that $(X, s_3s_2s_1, t_3t_2t_1)$ is an amalgam of (A, f, B, g, C).

So by symmetry we have to show that the diagram (A, f, B, f', B') has an amalgam (D, s_1, t). We use Lemma 0.2.4. We have $B = M_1^{I_1} \times \ldots \times M_n^{I_n}$ and $B' = M_{z_1}^{J_{z_1}} \times \ldots \times M_{z_y}^{J_{z_y}}$ where $y \leq n$ and $J_{z_1} \subset I_{z_1}, \ldots, J_{z_y} \subset I_{z_y}$. Now let $z \neq w \in B$ so that $\pi(z) \neq \pi(w)$ for some projection π on B. Then πf is a homomorphism of A into some member of $\mathscr{V}_{MI}$ so that by the assumption of the corollary there is a homomorphism h on B' into the same member of $\mathscr{V}_{MI}$ such that $hf' = \pi f$.

Finally let $z \neq w \in B'$ and let π be a projection on B' such that $\pi(z) \neq \pi(w)$. Since f' is a subdirect representation, $\pi f'$ is an epimorphism of A onto some member of $\mathscr{V}_{MI}$. It follows from Lemma 1.1.1 that there is an epimorphism h on B onto the same member of $\mathscr{V}_{MI}$ such that $hf = \pi f'$. This completes the proof.

1.2 On extension of $\underline{2}$–congruences of members of $\mathscr{N}$.

One of the main properties of members of $\mathrm{Amal}(\mathscr{N})$ is described in the following simple

Lemma 1.2.1. Let $A \in \mathrm{Amal}(\mathscr{N})$ and $A \leq B \in \mathscr{N}$. Then every $\underline{2}$–congruence on A can be extended to a $\underline{2}$–congruence on B.

Proof: Let $g\colon A \to N$ be a homomorphism with $g(A) \cong \underline{2}$ and $g(A) \neq \{a,c\}$ (see Figure 1). By Theorem 0.2.2 (2) there is $h\colon B \to N$ such that $h(A) = g(A)$. By Corollary 0.1.2 there is a retraction k from h(A) onto g(A) so that the kernel of kh is a $\underline{2}$–congruence on B which extends kerg.

Lemma 1.2.2. Let $A \leq B \in \mathscr{N}$. Then the following are equivalent:

(1) Every $\underline{2}$–congruence on A can be extended to a $\underline{2}$–congruence on B.

(2) If θ is a distributive congruence on A and u/v is an N–quotient of B then $\mathrm{Con}(u,v)|_A \subset \theta$.

Proof: Let ψ_A and ψ_B be the smallest distributive congruences on A and B respectively. Consider the following sets:

X_B – the set of all $\underline{2}$–congruences on B

$X_A = \{\varphi \in X_B\colon\ \varphi|_A \text{ is a } \underline{2}\text{–congruence on A}\}$

T_B = the set of all N–quotients of B

T_A = the set of all N–quotients of A.

Since $T_A \subset T_B$ we have the following in the lattice Con(B):

$$\sum_{u/_v \in T_A} \mathrm{Con}(u,v) \subset \sum_{u/_v \in T_B} \mathrm{Con}(u,v).$$

Thus we have $\psi_A = \sum_{u/_v \in T_A} (\mathrm{Con}(u,v)|_A) \subset \sum_{u/_v \in T_B} (\mathrm{Con}(u,v)|_A) = \psi_B|_A$.

Now (1) holds iff $\psi_A = (\cap X_A)|_A$.

Further, since $X_A \subset X_B$ we have

$(\cap X_A)|_A \supset (\cap X_B)|_A = \psi_B|_A = (\sum_{u/_v \in T_B} \mathrm{Con}(u,v))|_A$.

Thus (1) holds iff $\psi_A = (\sum_{u/_v \in T_B} \mathrm{Con}(u,v))|_A$ iff (2) holds.

Corollary 1.2.3. Let f: $A \to B \in \mathscr{N}$ be an embedding and assume that every $\underline{2}$-congruence on f(A) can be extended to a $\underline{2}$-congruence on B. Then if C is a finite Boolean algebra and g: $A \to C$ is an epimorphism, there is an epimorphism h: $B \to C$ such that g = hf.

Corollary 1.2.4. Let N^I be a direct power of the pentagon and let f: $A \to N^I$ be an embedding. Assume that every $\underline{2}$-congruence on f(A) can be extended to a $\underline{2}$-congruence on N^I. If g: $A \to N$ is a homomorphism such that $g(A) \cong \underline{3}$ and the critical quotient of N is a sublattice of g(A) then there is no h: $N^I \to N$ such that g = hf.

Proof: It follows from Lemma 1.2.2 that if such h existed, then $h(N^I)$ would be a distributive sublattice of N. However, since the critical quotient of N is a sublattice of g(A), the lattice $h(N^I)$ must be a chain having at least three elements. This is impossible since every distributive image of N^I is a Boolean algebra (see also [2] Theorem 7).

Our next result is concerned with restrictions of $\underline{2}$-congruences of a product of members of $\mathscr{N}$ into its factors.

Lemma 1.2.5. Let $A = \prod_{\gamma \in \alpha} A_\gamma$ be a product of members of $\mathscr{N}$. If θ is a $\underline{2}$-congruence on A then for some $\gamma \in \alpha$ there is an embedding $A_\gamma \leq A$ such that $\theta|_{A_\gamma}$ is a $\underline{2}$-congruence on A_γ.

Proof: Let $\{u > v\} \subset A$ such that $A/\theta = \{u/\theta, v/\theta\}$ and let

$$\Omega = \{\beta \in \alpha \colon u_\beta > v_\beta\}.$$

(Here u_β and v_β are the β^{th} coordinates of u and v respectively).

Consider two possible cases:

Case 1: for some $\gamma \in \Omega$ the set $\{u_\gamma > v_\gamma\} \subset A_\gamma$ is a $\underline{2}$-subset of A_γ.

Let f: $A_\gamma \to \underline{2} = \{0,1\}$ be an epimorphism with $f(u_\gamma) = 1$ and $f(v_\gamma) = 0$. Consider an embedding $A_\gamma \leq A$ given by

$$x \to \bar{x} \quad (x \in A_\gamma, \bar{x} \in A)$$

which can be described as follows:

(1) $\bar{x}_\gamma = x$

(2) If $\beta \in \Omega$ and $\beta \neq \gamma$ then

$\bar{x}_\beta = u_\beta$ if f(x) = 1 and

$\bar{x}_\beta = v_\beta$ if f(x) = 0.

(3) If $\beta \in (\alpha - \Omega)$ then $\bar{x}_\beta = u_\beta = v_\beta$.

It follows from (1), (2) and (3) that $\{u>v\} \subset A_\gamma \leq A$ so that $\theta|_{A_\gamma}$ is a $\underline{2}$-congruence on A_γ.

Case 2: Not Case 1, i.e. for any $\beta \in \Omega$ the set $\{u_\beta > v_\beta\} \subset A_\beta$ is not a $\underline{2}$-subset of A_β.

Since θ is completely meet irreducible in Con(A), by Jónsson's Lemma there is a filtral congruence ψ on A such that $\psi \subset \theta$. Let $\mathscr{D}$ be an ultrafilter on α which induces ψ. We have $\Omega \in \mathscr{D}$. Furthermore it follows from the assumption which is made in Case 2 that, for any $\beta \in \Omega$, the set $\{\beta\}$ does not belong to $\mathscr{D}$. Thus $(\Omega - \{\beta\}) \in \mathscr{D}$ for $\beta \in \Omega$. Choose any $\gamma \in \Omega$. Since $\{u_\gamma > v_\gamma\} \subset A_\gamma$, the lattice A_γ is non-trivial and, since $A_\gamma \in \mathscr{N}$, there is at least one epimorphism f: $A_\gamma \to 2 = \{0,1\}$. Consider an embedding $A_\gamma < A$ associated with f which is similar to that of Case 1. Let $\{x > y\} \subset A_\gamma$ be such that f(x) = 1 and f(y) = 0. It follows that $u/\theta = \bar{x}/\theta$ and $v/\theta = \bar{y}/\theta$ so that $\theta|_{A_\gamma}$ is a $\underline{2}$-congruence on A_γ.

For our next result we assume that B is a subdirect power of N and B is an image of $A \in \mathrm{Amal}(\mathscr{N})$. Let $\Omega \in \mathrm{Con}(A)$ be such that $A/\Omega \cong B$. For $a \in A$ we let $\bar{a} \in B$ be an image of a/Ω under the isomorphism between A/Ω and B. Also if $\theta \in \mathrm{Con}(A)$ with $\Omega \leq \theta$ we let $\bar{\theta} \in \mathrm{Con}(B)$ be the image of θ under the isomorphism between the principal filter in Con(A) generated by Ω and the lattice Con(B).

Proposition 1.2.6. Let $B \leq N^I$ be a regular subdirect representation. If $\bar{\theta}$ is a $\underline{2}$–congruence on B then $\bar{\theta}$ can be extended to a $\underline{2}$–congruence on N^I.

Let $\{\varphi_s\colon s \in S\}$ be the set of all N–congruences on A, so that each φ_s is a kernel of an epimorphism of A onto N. Since the subdirect representation $B \leq N^I$ in Proposition 1.2.6 is regular we may assume that $I \subset S$. Let $J = S{-}I$ so that $N^S = N^I \times N^J$.

Consider a subdirect representation $A \leq N^S$ such that for $a \in A$:

$$a_i = \bar{a}_i \ \text{ if } i \in I$$
$$a_j = (a/\varphi_j) \in N \ \text{ if } j \in J.$$

Let $\bar{\theta} \in \mathrm{Con}(B)$ as in Proposition 1.2.6 and consider $\{\bar{u} > \bar{v}\} \subset B$ such that $B/\bar{\theta} = \{\bar{u}/\bar{\theta}, \bar{v}/\bar{\theta}\}$. Then for $\{u > v\} \subset A$ we have $A/\theta = \{u/\theta, v/\theta\}$. Since $A \in \mathrm{Amal}(\mathscr{N})$, by Lemma 1.2.1 θ can be extended to a $\underline{2}$–congruence Σ on N^S. By Jónsson's Lemma there is a filtral congruence ψ on N^S such that $\psi \subset \Sigma$. Let $\mathscr{D}$ be an ultrafilter on S which induces ψ. Since $N^S/\psi \cong N$ and $N^S/\Sigma = \{u/\Sigma, v/\Sigma\}$ we have that for some $\{p,q\} \neq \{a,c\}$ (see Figure 1) the set $X = \{s \in S\colon u_s = p > q = v_s\}$ belongs to $\mathscr{D}$. Now if $(X \cap I) \in \mathscr{D}$ then Proposition 1.2.6 holds at once since, for $i \in I$, u and v have the same i^{th} coordinates as $\bar{u}$ and $\bar{v}$. So assume that $(X \cap J) = Y \in \mathscr{D}$. In order to prove Proposition 1.2.6 under this assumption we need a lemma.

Lemma 1.2.7. (1) There is an embedding $f\colon B \to N^S$ such that $\Sigma|_{f(B)} = \bar{\theta}$.

(2) There is an embedding $g\colon N^I \to N^S$ such that $\Sigma|_{g(N^I)} = \Delta$ is a $\underline{2}$–congruence on N^I.

(3) Let $h\colon N^I \to N^S$ be an embedding such that for $y, z \in N^I$

$$[h(y)]_i = y_i \ \text{ for } i \in I$$
$$[h(y)]_j = [h(z)]_j \ \text{ for } j \in J.$$

Then there is a $\underline{2}$–congruence Σ' on N^S such that $\Sigma'|_{h(N^I)} = \Delta$ where Δ is as in (2).

Proof: (1) Let $t\colon B \to \underline{2} = \{0, 1\}$ be an epimorphism with $\ker t = \bar{\theta}$. Then f can be described as follows:

$[f(\bar{a})]_i = \bar{a} = a_i$ for $\bar{a} \in B$ and $i \in I$.

For $y \in Y$, $[f(\bar{a})]_y = p = u_y$ if $t(\bar{a}) = 1$ and

$[f(\bar{a})]_y = q = v_y$ if $t(\bar{a}) = 0$.

For $s \in S - (I \cup Y)$ and $\bar{a}, \bar{b} \in B$ $[f(\bar{a})]_s = [f(\bar{b})]_s$.

(2) The idea here is similar to the one in the proof of (1). Consider any epimorphism $t\colon N^I \to \underline{2} = \{0,1\}$. Then g is described as follows:

$[g(k)]_i = k_i$ for $k \in N^I$ and $i \in I$.

For $y \in Y$, $[g(k)]_y = p = u_y$ if $t(k) = 1$ and

$[g(k)]_y = q = v_y$ if $t(k) = 0$.

For $s \in S - (I \cup Y)$ and $k, l \in N^I$, $[g(k)]_s = [g(l)]_s$.

(3) Let $h\colon N^I \to N^S$ be as in (3) and $g\colon N^I \to N^S$ be as in (2). Since $N^I \in \mathrm{Amal}(\mathscr{N})$ (see Theorem 0.2.2 (3) and Example 0.2.1), the diagram (N^I, g, h, N^S) has an amalgam (Q, g_0, h_0) with $Q \in \mathscr{N}$. (see Figure 5). Since $N^S \in \mathrm{Amal}(\mathscr{N})$, by Lemma 1.2.1, Σ can be extended to a $\underline{2}$–congruence Π on Q. Letting $\Sigma' = \Pi|_{h_0(N^S)}$, we have $\Sigma'|_{h_0 h(N^I)} = \Delta$.

Figure 5

Proof of Proposition 1.2.6.

As we noticed before we only need to consider the case $(X \cap J) = Y \in \mathscr{D}$. Let f and h be as in Lemma 1.2.7 (1) and (3). Since $B \leq N^I$ and h: $N^I \to N^S$ we an consider a diagram (B, f, h, N^S). Since the variety of all lattices satisfies the amalgamation property (see [8]), the above diagram has an amalgam (R, f_0, h_0) where R is a lattice not necessarily in $\mathscr{N}$. (see Figure 6). The proposition follows now from Lemma 1.2.7 since $\bar{\partial} = \Sigma|_{f_0 f(B)} = \Sigma'|_{hh_0(B)} = \Delta|_{h_0 h(B)}$.

Figure 6

2. MEMBERS OF AMAL($\mathscr{N}$)

In this section we consider the non–trivial members of Amal($\mathscr{N}$). The following characterization theorem can be found in [9].

Theorem 2.1. (1) (Bergman). A lattice $A \in \mathscr{N}$ belongs to Amal($\mathscr{N}$) iff A is a congruence extensile, subdirect power of N and for any subdirect representation f: $A \to N^I$ and any homomorphism g: $A \to N$ there is a homomorphism h: $N^I \to N$ such that $g = hf$.

(2) A finite lattice $A \in \mathscr{N}$ belongs to Amal($\mathscr{N}$) iff A is a subdirect power of N and does not have $\underline{3}$ as a homomorphic image.

Our next theorem gives a characterization of membership of Amal($\mathscr{N}$) which is different from that in Theorem 2.1 (1). This characterization is crucial for the proof of Theorem 3.1:

Theorem 2.2. For $A \in \mathscr{N}$, the following are equivalent:

(1) $A \in$ Amal($\mathscr{N}$).

(2) If $A \leq B \in \mathscr{N}$, then every $\underline{2}$–congruence on A can be extended to a $\underline{2}$–congruence on B and $\underline{3}$ is not an image of A.

(3) A is a subdirect power of N and for any regular subdirect representation f: $A \to N^I$ and any homomorphism g: $A \to N$ there is a homomorphism h: $N^I \to N$ such that $g = hf$.

(4) A is a subdirect power of N, $\underline{3}$ is not an image of A and if f: $A \to N^I$ is a regular subdirect representation then any $\underline{2}$–congruence on A can be extended to a $\underline{2}$–congruence on N^I.

Proof: (1) ⇒ (2). If $A \in$ Amal($\mathscr{N}$) and $A \leq B \in \mathscr{N}$ then by Lemma 1.2.1 every $\underline{2}$–congruence on A can be extended to a $\underline{2}$–congruence on B. Further, Theorem 0.2.2 (2) and Corollary 1.2.4 jointly imply that $\underline{3}$ is not an image of A.

(2) ⇒ (1). If (2) holds then to show that (1) holds we again use Theorem 0.2.2 (2). Let g: $A \to N$ be a homomorphism and f: $A \to B \in \mathscr{N}$ be an embedding.

Since $\underline{3}$ is not an image of A we have that g(A) must be isomorphic to one of the following lattices: $\underline{1}$, N, $\underline{2}$ or $\underline{2} \times \underline{2}$. For non–trivial cases: If $g(A) \cong N$ then the existence of h follows from Lemma 1.1.1. If $g(A) \cong \underline{2}$ of $g(A) \cong \underline{2} \times \underline{2}$ the existence of h follows from Corollary 1.2.3.

(1) ⇒ (3) follows from Theorem 2.1 (1).

Now by Corollary 1.2.4, (3) implies (4).

(4) ⇒ (1). By Corollary 1.1.3 we have to show that if $g\colon A \to N$ is a homomorphism then there is $h\colon N^I \to N$ such that $hf = g$. By assumption the non–trivial images of A under g are N, $\underline{2}$ or $\underline{2} \times \underline{2}$, so that one can use arguments similar to those in the proof (2) ⇒ (1).

Corollary 2.3. If $A \in \mathrm{Amal}(\mathscr{N})$ then every finite distributive image of A is a Boolean algebra.

Proof: The corollary follows from Theorem 2.2 since $\underline{3}$ is not an image of A.

We are now going to consider certain extensions of members of $\mathscr{N}$ which belong to $\mathrm{Amal}(\mathscr{N})$ (see Proposition 2.5).

We first need the following

Lemma 2.4. Let θ_0, θ_1 be $\underline{2}$–congruences on $B \in \mathscr{N}$ with $\underline{2}$–sublattices $\{u > v\}$ and $\{w > z\}$ of B such that $B/\theta_0 = \{u/\theta_0, v/\theta_0\}$ and $B/\theta_1 = \{w/\theta_1, z/\theta_1\}$. Assume the following: For some $A \in \mathrm{Amal}(\mathscr{N})$ and embeddings $f_0, f_1\colon A \to B$, the restrictions $f_0(A)|_{\theta_0}$ and $f_1(A)|_{\theta_1}$ are distinct $\underline{2}$–congruences on A. Then $\varphi = \theta_0 \cap \theta_1$ is not a $\underline{3}$–congruence on B.

Proof: Suppose that $B/\varphi \cong \underline{3}$. Since $\underline{3}$ is a projective lattice we may assume that $v = w$ so that $B/\varphi = \{u/\varphi, v/\varphi, z/\varphi\}$. Then for $k \in B$ there is $t \in \{u, v, z\}$ such

that $(k, t) \in (\theta_0 \cap \theta_1)$. Since $\theta_0|_{f_0(A)}$ and $\theta_1|_{f_1(A)}$ are distinct $\underline{2}$ congruences on A without loss of generality we may assume that $\{u, v\} \subset f_0(A)$ and $\{v, z\} \subset f_1(A)$. Let $D \in \mathscr{N}$ be an amalgam of the diagram (A, f_0, f_1, B). By Theorem 2.2 (2) θ_0, θ_1 can be extended to $\underline{2}$–congruences $\overline{\theta}_0$, $\overline{\theta}_1$ on D. Let $\overline{\varphi} = \overline{\theta}_0 \cap \overline{\theta}_1$. Then $\overline{\varphi}$ restricted on $f_0(A) = f_1(A)$ is a $\underline{3}$–congruence on A which contradicts Theorem 2.2.

As a corollary we have the following

Proposition 2.5. Let $B \in \mathscr{N}$, and assume that, for any distinct $\underline{2}$–congruences θ_0 and θ_1 on B there is $A \in \mathrm{Amal}(\mathscr{N})$ and embeddings $f_0, f_1\colon A \to B$ such that $f_0(A)|_{\theta_0}$ and $f_1(A)|_{\theta_1}$ are distinct $\underline{2}$–congruences on A. Then $B \in \mathrm{Amal}(\mathscr{N})$.

Proof: It follows from the given conditions that if $B \leq C \in \mathscr{N}$ then every $\underline{2}$–congruence on B can be extended to a $\underline{2}$–congruence on C. Further from Lemma 2.4, B does not have $\underline{3}$–congruences. The proposition now follows from Theorem 2.2.

3. THE MAIN THEOREM

The principal result of this paper is the following

Theorem 3.1. $\mathrm{Amal}(\mathscr{N})$ is an elementary class. It is closed under reduced products and therefore is determined by Horn sentences. Furthermore, if B is an image of $A \in \mathrm{Amal}(\mathscr{N})$ and B is a subdirect power of the pentagon then $B \in \mathrm{Amal}(\mathscr{N})$.

Proof: We first prove the last statement of the theorem. Let B be a subdirect power of N and assume that B is an image of $A \in \mathrm{Amal}(\mathscr{N})$. It follows from Proposition 1.2.6 that if $B \leq N^I$ is any regular subdirect representation then any $\underline{2}$–congruence on B can be extended to a $\underline{2}$–congruence on N^I. Since $A \in \mathrm{Amal}(\mathscr{N})$, the lattice $\underline{3}$ is not an image of A and therefore $\underline{3}$ cannot be an image of B. Thus

$B \in \mathrm{Amal}(\mathscr{N})$ by Theorem 2.2 (2).

Next we show that $\mathrm{Amal}(\mathscr{N})$ is closed under direct products. Let $A = \prod_{\gamma \in \alpha} A_\gamma$ be a direct product of members of $\mathrm{Amal}(\mathscr{N})$. We use Proposition 2.5, that is we prove that $A \in \mathrm{Amal}(\mathscr{N})$ by showing that for any distinct congruences θ_0, θ_1 on A there is $\gamma \in \alpha$ and embeddings $f_0, f_1\colon A_\gamma \to A$ such that $\theta_0|_{f_0(A_\gamma)}$ and $\theta_1|_{f_1(A_\gamma)}$ are distinct $\underline{2}$–congruences on A_γ. Now if θ_0 and θ_1 are distinct congruences on A then $A/(\theta_0 \cap \theta_1)$ is isomorphic either to $\underline{\mathbf{3}}$ or $\underline{\mathbf{2}} \times \underline{\mathbf{2}}$. In either case we have $u, v, z \in A$ with $u > v > z$ such that

$(u,v) \notin \theta_0$, $(v,z) \in \theta_1$ and

$(v,z) \notin \theta_0$, $(u,v) \in \theta_0$.

By Jónsson's Lemma there are filtral congruences φ_0, φ_1 on A such that $\varphi_0 \subset \theta_0$ and $\varphi_1 \subset \theta_1$. Let $\mathscr{D}_0$ and $\mathscr{D}_1$ be ultrafilters which induce φ_0 and φ_1. We have

$R = \{\beta \in \alpha\colon u_\beta > v_\beta\} \in \mathscr{D}_0$

$S = \{\beta \in \alpha\colon v_\beta > z_\beta\} \in \mathscr{D}_1$.

Since $u > v > z$ we have $R \cap S \neq \varphi$. Pick $\gamma \in (R \cap S)$. There are three possible cases:

(1) The set $\{\gamma\}$ belongs to both $\mathscr{D}_0$ and $\mathscr{D}_1$

(2) $\{\gamma\}$ belongs to neither $\mathscr{D}_0$ nor $\mathscr{D}_1$.

(3) $\{\gamma\}$ belongs to one ultrafilter and not the other.

If (1) holds then f_0, f_1 can be any embeddings satisfying:

$$[f_0(x)]_\gamma = x = [f_1(x)]_\gamma \quad \text{for all } x \in A_\gamma.$$

Suppose (2) holds. Then $(R - \{\gamma\}) \in \mathscr{D}_0$ and $(S - \{\gamma\}) \in \mathscr{D}_1$. Since A_γ is a subdirect power of N, there are at least two distinct epimorphisms $k, t\colon A_\gamma \to \underline{2} = \{0,1\}$. Thus one can define embeddings f_0, f_1 which are associated with k and t similar to those in the proof of Case 1 of Lemma 1.2.6. The case (3) is a combination of (1) and (2).

Thus we have shown that every direct product of members of $\mathrm{Amal}(\mathscr{N})$ belongs to $\mathrm{Amal}(\mathscr{N})$. Now if B is a reduced product of members of $\mathrm{Amal}(\mathscr{N})$ then B must be a subdirect power of N (see Lemma 0.1.9). On the other hand B is an image of a product A of members of $\mathrm{Amal}(\mathscr{N})$. Since $A \in \mathrm{Amal}(\mathscr{N})$ it follows that $B \in \mathrm{Amal}(\mathscr{N})$. In particular every ultraproduct of members of $\mathrm{Amal}(\mathscr{N})$ belongs to $\mathrm{Amal}(\mathscr{N})$ so that $\mathrm{Amal}(\mathscr{N})$ is an elementary class (see [12]). It is determined by Horn sentences since it is closed under reduced products (see [4]).

4. EXAMPLES OF VARIETIES WHOSE AMALGAMATION CLASS CONTAINS Amal($\mathscr{N}$) AS A SUBCLASS

The following result gives examples of varieties as indicated in the title above. The varieties considered here are by no means the only ones satisfying this property.

Let $\mathscr{V}$ be a variety generated by two maximal subdirectly irreducible members M and K where M is a finite modular lattice and K is either N or a lattice from the set $X = \{L_1, L_2\} \cup \{L_{15}^{n}\colon n \in \omega\}$.

Proposition 4.1. Amal($\mathscr{N}$) $\subset$ Amal($\mathscr{V}$). In particular if $M = \underline{1}$ or $M = \underline{2}$ then Amal($\mathscr{N}$) is a subclass of the amalgamation class of the variety generated by a lattice from the set X.

For Lemmas 4.2 – 4.4 we assume that $L \in X$.

Lemma 4.2. Let $A \in \mathrm{Amal}(\mathscr{N})$ and let $g\colon A \to K$ be a homomorphism.

(1) If $K = L$ then $g(A) \cong N$ or $g(A) \cong \underline{2}^m$ for some $m \in \omega$.

(2) If K is a finite modular lattice then $g(A) \cong \underline{2}^m$ for some $m \in \omega$.

Proof: If g(A) is a non–distributive sublattice of L_i (i = 1, 2) then $g(A) \cong N$ for otherwise $g(A) = L_i$. If g(A) is a non–distributive sublattice of .L_{15}^{n} then $g(A) \cong N$ for otherwise g(A) will have $\underline{3}$ as its image which is impossible by Theorem 2.2. Now if g(A) is a distributive sublattice of L, so that $g(A) \cong \underline{2}^m$ by Corollary 2.3. (2) follows from Corollary 2.3 since g(A) is a distributive sublattice of K.

The proof of the next lemma is trivial and left for the reader.

Lemma 4.3. (1) Let P_1 and P_2 be distinct pentagons in L such that $b_1 \in P_1$ and $b_2 \in P_2$ (see Figures 2 and 3). Then the map $b_1 \to b_2$ can be extended to an automorphism α: $L \to L$ such that $\alpha(P_1) = P_2$.

(2) Let {u>v} be a $\underline{2}$–subset of L. Then {u,v} can be chosen so that it is a $\underline{2}$–subset of some pentagon P in L. Furthermore there is a co–retraction β: $L \to \{u,v\}$ so that $\beta(L) = \beta(P)$.

Lemma 4.4. Let K = N or K = L and let M be a finite modular lattice. Assume that $A \leq B = K^I \times M^J$ where $A \in \mathrm{Amal}(\mathscr{N})$.

(1) If θ is an N–congruence on A then θ can be extended to an L–congruence on B

(2) If ψ is a $\underline{2}$–congruence on A then ψ can be extended to a $\underline{2}$–congruence on B.

Proof: (1) By Jónsson's Lemma there is a filtral congruence φ on B such that $\varphi|_A \subset \theta$. Furthermore if $\mathscr{D}$ is an ultrafilter on $I \cup J$ which induces φ then clearly $I \in \mathscr{D}$ so that φ is a K–congruence on B.

(2) It is sufficient to consider only K = L. Since A is a subdirect power of N there is an N–congruence θ on A with $\theta \subset \psi$. Let φ be as in (1) so that $\varphi|_A \subset \theta \subset \psi$. Let h: $B \to L$ be a homomorphism such that $\varphi = \ker h$. Then $\ker(\beta h)$ where β is as in Lemma 4.3 (2) is a $\underline{2}$–congruence on B which extends ψ.

Proof of Proposition 4.1

By assumption $\mathscr{V}$ has at most two maximal subdirectly irreducible members M and K where M is a modular lattice and K is either N or K is a lattice from the set X. Let $A \in \mathrm{Amal}(\mathscr{N})$. Since every member of $\mathscr{V}$ is embeddable into $B = K^I \times M^J$ for some index sets I and J, it is sufficient to show, by Theorem 0.2.2 (2), that if f: $A \to B$ is an embedding and g is a homomorphism from A into K or M then there is a homomorphism h from B into K or M such that g = hf.

As before it is sufficient to consider only K = L. By Lemma 4.4 we have either $g(A) \cong N$ or $g(A) \cong \underline{2}^m$ for some $m \in \omega$. Let $g(A) \cong N$ and let p: $B \to L$ be an epimorphism such that kerp is an extension of kerg (see Lemma 4.4 (1)). Then h is either p or αp where αR $L \to L$ is as in Lemma 4.3 (1). Now assume that $g(A) \cong \underline{2}^m$. The case m = 0 is trivial and Lemma 4.4 (2) takes care of the case m = 1. If m > 1 then we apply the case m = 1 m times. Finally if g: $A \to M$ then $g(A) \cong \underline{2}^m$ so that in this case the proof is the same as for g: $A \to K$ with $g(A) \cong \underline{2}^m$.

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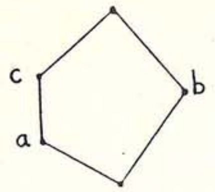

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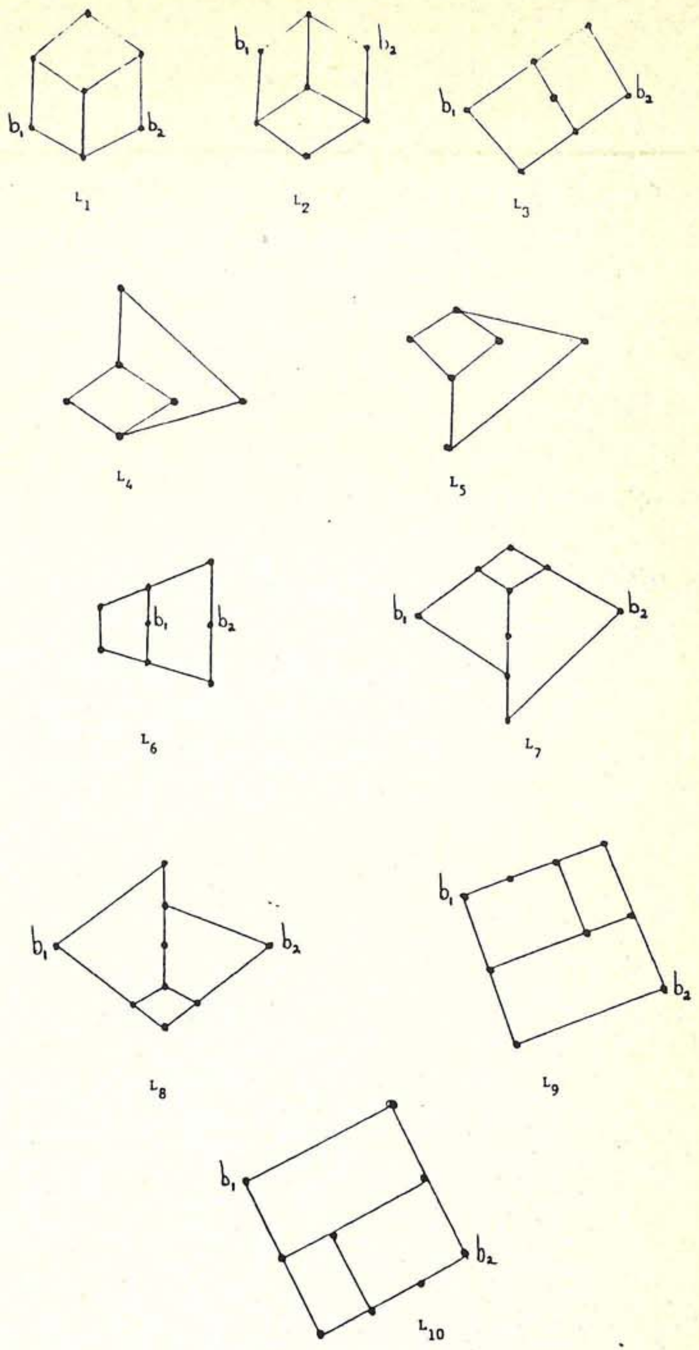
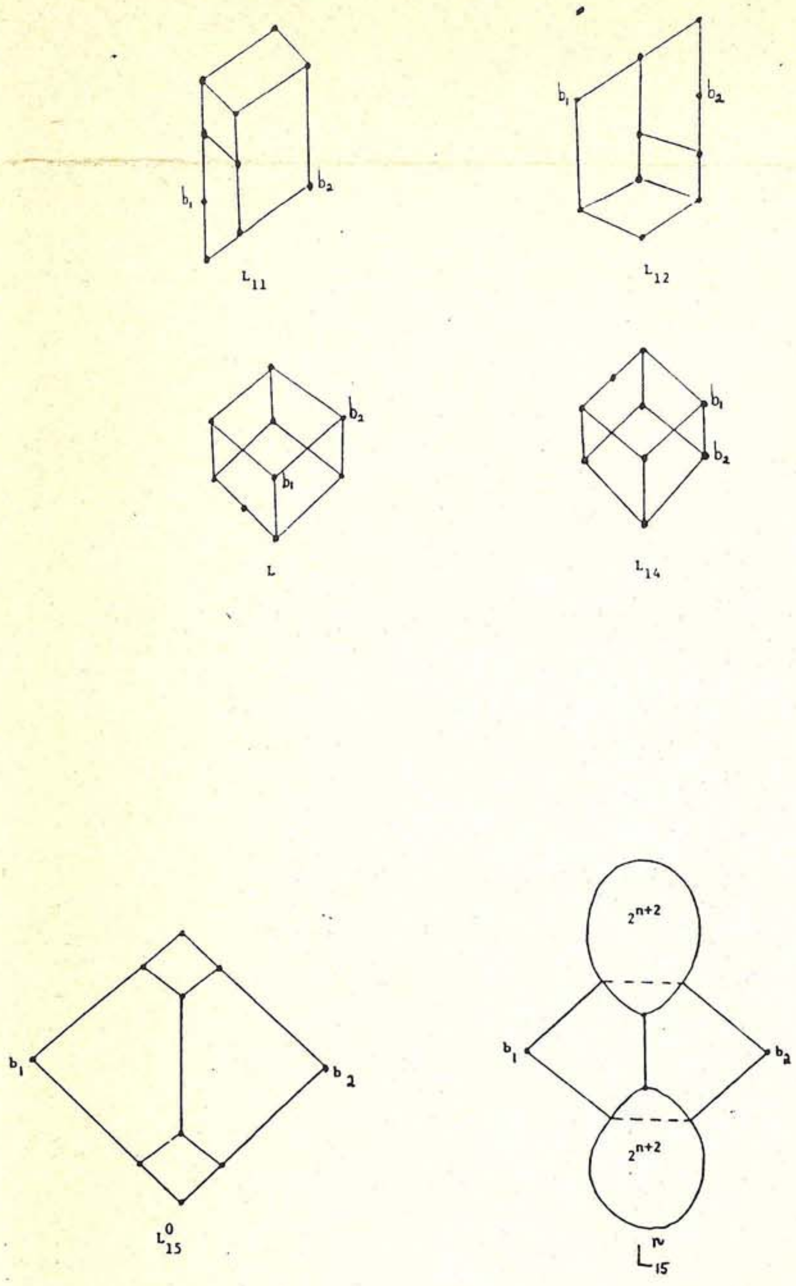
Figure 1.



L_1 L_2 L_3 L_4 L_5 L_6 L_7 L_8 L_9 L_{10}

Figure 2.



L_{11} L_{12} L L_{14} L_{15}^0 $L_{15}^{\sim}$

Figure 3.

$$\begin{tikzcd}
 & & & X & & & \\
 & & G \arrow[ur, "l_3"] & & F \arrow[ul, "s_3"'] & & \\
 & & & H \arrow[ul, "u"'] \arrow[ur, "v"] & & & \\
 & E \arrow[uur, "l_2"] & & & & D \arrow[uul, "s_2"'] & \\
C \arrow[ur, "l_1"] & & C' \arrow[ul, "m"'] \arrow[uur, "p"] & & B' \arrow[uul, "q"'] \arrow[ur, "t"] & & B \arrow[ul, "s_1"'] \\
 & & & A \arrow[ulll, "g"] \arrow[ul, "g'"] \arrow[ur, "f'"'] \arrow[urrr, "f"'] & & &
\end{tikzcd}$$

Figure 4

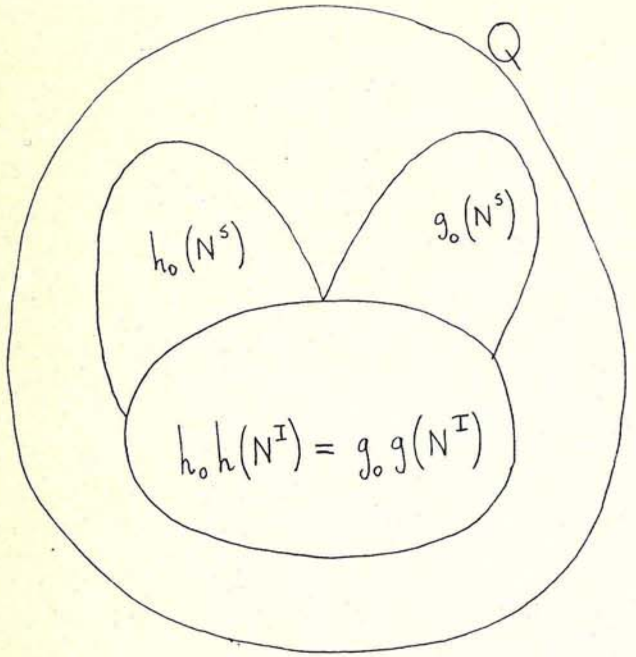


Figure 5.

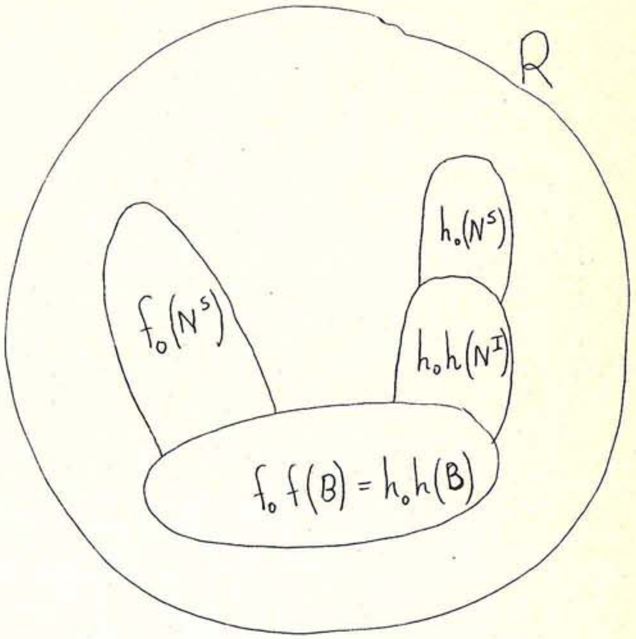


Figure 6.